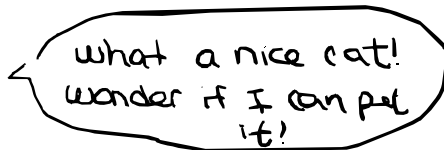


Deligne Categories $\text{Rep}(S_t)$, $\text{Rep}(GL_t)$

Big problem:

$\text{Rep}(G)$

G alg group



Definition: A ~~nice~~ pre-tannakian category is a category which satisfies the same nice properties as $\text{Rep}(G)$

Motivation: Key words: Grothendieck, motives

↳ read intro 'Tannakian categories' Milne.

Natural Question: Are all pre-tannakian categories $\text{Rep}(G)$ for some G ?

1) There is a simple restriction, which is, having moderate growth:

part of 'nice' is being abelian

a) simple objects make sense in this context

(think: have no non-trivial subobjects)

b) length of an object (in the sense of Jordan-Hölder) makes sense too

$$\hookrightarrow \ln(X^{\otimes n}) \leq \dim(X^{\otimes n}) = \dim(X)^n \leadsto \text{grows} \leq \text{exp with } n$$

Deligne's thm: Converse holds in char 0 $\rightarrow$ any pre-Tann cat w/ moderate growth is $\text{Rep}(G)$

Rmk: There are known counterexamples in char $p > 0$
 $\hookrightarrow$ "semi-simplification of $\text{Rep}(\mathbb{Z}/p\mathbb{Z})$ ", called Ver_p
 $p=2 \rightarrow \text{Vec}$, $p=3 \rightarrow \text{sVec}$, $p=5 \rightarrow$ counterexample.

New natural question: can we find pre-Tann cat of non-moderate growth?

Properties we will focus on:

$\hookrightarrow$ Has nice $\otimes$ product

$\hookrightarrow$ Is rigid $\sim$ objects have duals

From VS: $X \otimes X^* \xrightarrow{\text{ev}} k$ $V \otimes \lambda \rightarrow \lambda(V)$

$X \otimes X^* \leftrightarrow \text{Matrices/End}(X)$ (put $e_i \otimes e_j^*$ in (i,j))

We also have a copy of k in $X \otimes X^*$ w/ $k \text{Id}$

That's the coev map: $k \xrightarrow{\text{cv}} X^* \otimes X$

$\text{Id} \cdot X = X$ becomes

$X \xrightarrow{1 \otimes \text{cv}} X \otimes X^* \otimes X \xrightarrow{\text{ev} \otimes 1} X$ also $X^* \xrightarrow{1 \otimes \text{cv}} X^* \otimes X \otimes X^* \xrightarrow{\text{ev} \otimes 1} X^*$
"Id"

This will be the categorical def of dual: X^* with maps ev, coev $X \otimes X^* \xrightarrow{\text{ev}} k$, $k \xrightarrow{\text{coev}} X \otimes X^*$ such that
 $X \xrightarrow{1 \otimes \text{cv}} X \otimes X^* \otimes X \xrightarrow{\text{ev} \otimes 1} X$, $X^* \xrightarrow{1 \otimes \text{cv}} X^* \otimes X \otimes X^* \xrightarrow{\text{ev} \otimes 1} X^*$.

We can use this to define trace/dim!

Fix a basis.

$$\text{ev}: e_i^* \otimes e_j = \delta_{ij} \quad \text{coev}: 1 \mapsto \sum e_i^* \otimes e_i$$

In matrices:

$$\text{ev}(A) \stackrel{''}{=} \sum a_{ij} e_i^* \otimes e_j = \sum a_{ii} e_i^* \otimes e_i = \text{tr}(A)$$

$$\text{coev}(1) = \sum e_i^* \otimes e_i \stackrel{''}{=} \text{Id}$$

$$\text{So, given } f: X \rightarrow X: \quad 1 \xrightarrow{\text{coev}} X^* \otimes X \xrightarrow{1 \otimes f} X^* \otimes X \xrightarrow{\text{ev}} 1$$

'get identity matrix' 'turn it into A' 'take the trace'

The composed map $\in \text{End}_K(1)$ is $\text{tr}(f)$

↳ if we ask $\geq k$, $\text{tr}(f) \in k$.

$$\text{Dim}(X) = \text{tr}(1_X) \text{ defined similarly: } 1 \xrightarrow{\text{coev}} X^* \otimes X \xrightarrow{\text{ev}} 1.$$

'put 1's in the diagonal, then take the trace'

————— " —————

Rep(S_t) $\rightsquigarrow$ We want to interpolate $\text{Rep}(S_n)$,

so we will try to 'talk about S_n without using anything specific to S_n , other than n '.

Then, if we replace the n by t , we can get ' $\text{Rep}(S_t)$ ', whatever that means.

key idea: Consider $V = \mathbb{C}^n$, with S_n acting by permuting coordinates. This is a faithful rep, so $g \mapsto \rho(g)$ is injective

(by rep theory sorcery) every irrep appears as a summand of $V^{\otimes m}$ for some m .

(Cat. theory sorcery) Karoubian envelope $\rightarrow$ we can take a category and artificially add all summands

key idea: $W = V \oplus \ker e \rightarrow \pi W \rightarrow V$ is idempotent

e idempotent $\rightarrow W = e(W) \oplus \ker(e)$

So, consider category with objects (W, e) and morphisms $(W, e) \rightarrow (V, f)$ given by $f \text{Hom}(W, V) e$

So, it is probably fine to study $V^{\otimes m} \longleftrightarrow [m]$ and maps between them, generalize to other t and then take the envelope!

key questions become: how to talk about maps $V^{\otimes m} \rightarrow V^{\otimes k}$? How to compose morphisms?

If we ignore the action, we have a basis for linear maps given by $e_i \mapsto f_j$ (put a 1 in (i, j) if seen as $\text{End}(X, Y)$ or $e_i \otimes e_j^*$ if seen as $X^* \otimes Y$). So, for example

$$(S_4), \quad V^{\otimes 3} \rightarrow V^{\otimes 4} \quad e_1 \otimes e_2 \otimes e_1 \rightarrow e_1 \otimes e_1 \otimes e_2 \otimes e_3$$

would be a map. However, $(3, 4)$. $e_1 \otimes e_2 \otimes e_1 = e_1 \otimes e_2 \otimes e_1$ should then be mapped to $(3, 4)$. $e_1 \otimes e_1 \otimes e_2 \otimes e_4$. So, really, the 'minimal building block' here would need to send $e_1 \otimes e_2 \otimes e_1$ to $e_1 \otimes e_1 \otimes e_2 \otimes e_3 + e_1 \otimes e_1 \otimes e_2 \otimes e_4$ to be S_4 -equivariant.

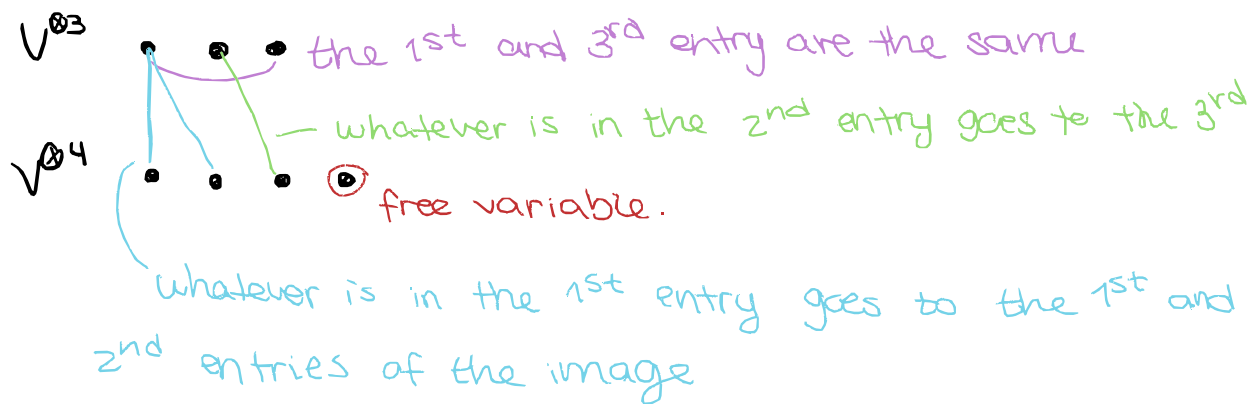
So, let's say we have the map

$$e_1 \otimes e_2 \otimes e_1 \rightarrow e_1 \otimes e_1 \otimes e_2 \otimes e_3 + e_1 \otimes e_1 \otimes e_2 \otimes e_4$$

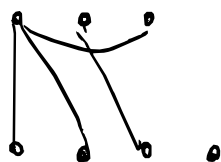
We can apply, say, $(1,3)$ to both sides:

$$e_3 \otimes e_2 \otimes e_3 \rightarrow e_3 \otimes e_3 \otimes e_2 \otimes e_1 + e_3 \otimes e_3 \otimes e_2 \otimes e_4$$

More generally, anything of the shape $e_j \otimes e_k \otimes e_j$ will be mapped to something determined by our map, via applying $(1,3)(2,k)$ on both sides. We can store this info as follows.



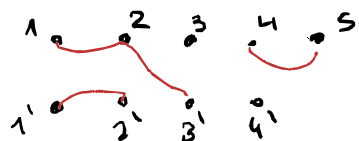
So, these diagrams store information regarding the relationship between the entries:



means $e_j \otimes e_k \otimes e_j \rightarrow \sum_{l \neq j, k} e_j \otimes e_j \otimes e_l \otimes e_l$
whenever $j \neq k$, 0 otherwise.

One can check these diagrams form a basis for $\text{Hom}_S(V^{\otimes m}, V^{\otimes k})$, indexed by partitions of $\{1, 2, \dots, m, 1', 2', 3', \dots, k'\}$.

Ex: $\{1, 2, 3, 4, 5, 1', 2', 3', 4'\}$ partition $\{1, 2, 3'\}, \{3', 4'\}, \{4, 5'\}, \{1', 2'\}$



If $i \neq j \neq k$, $e_i \otimes e_i \otimes e_j \otimes e_k \otimes e_k \mapsto \sum_{l \neq m \neq i} e_i \otimes e_i \otimes e_l \otimes e_l \otimes e_l$, otherwise 0.

How does composition work?

⚠ But I don't seem to be permissive here. - But I guess the idea is I could be.

$$e_1 \otimes e_2 \otimes e_1 \otimes e_3 \otimes e_4$$



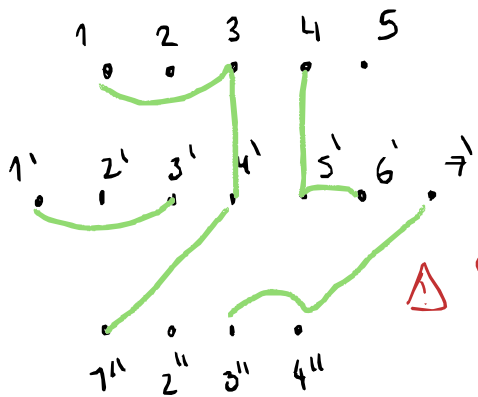
$$\sum e_i \otimes e_j \otimes e_k \otimes e_l \otimes e_m \otimes e_n$$

any i, j, k

number of connected components in the middle

$$\sum e_1 \otimes e_l \otimes e_k \otimes e_k$$

choices for i, j



$$e_1 \otimes e_2 \otimes e_1 \otimes e_3 \otimes e_4 \rightarrow \sum_{\substack{\text{any } x \\ \text{any } e}} e_1 \otimes e_k \otimes e_l \otimes e_l$$

why do we want to consider any i, j, k ?

↳ so we don't need to worry about cases, the variables all behave the same

why can we consider any i, j, k ?

↳ It's just a change of basis:

→ Before, have e_p connected to each partition

→ Now, have $v_p = \sum_{\tilde{p} \text{ more restrictive than } p} e_{\tilde{p}}$ ~ $\triangleright$ change of basis is if order by restrictiveness

So, to compose two diagrams, I get the map corresponding to following connections and eliminating the middle, multiplied by $n^{\# \text{connected components in the middle only}}$.

We are now ready to define $\text{Rep}(s, t)$:

1) The objects are given by $[n]$ $n \in \mathbb{N}$, represented by n dots corresponding to $V^{\otimes n}$

2) The morphisms are given by diagrams like above now abstracted

3) The composition is the same but with n replaced by t !

This has a tensor product! $V^{\otimes n} \otimes V^{\otimes m} = V^{\otimes n+m}$

↳ on objects, $[n] \otimes [m] = [n+m]$

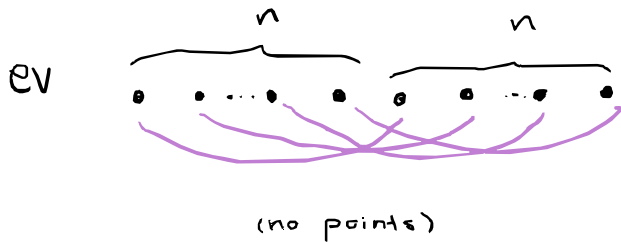
↳ on diagrams, just put them side by side!

↳ properties all work, trust!

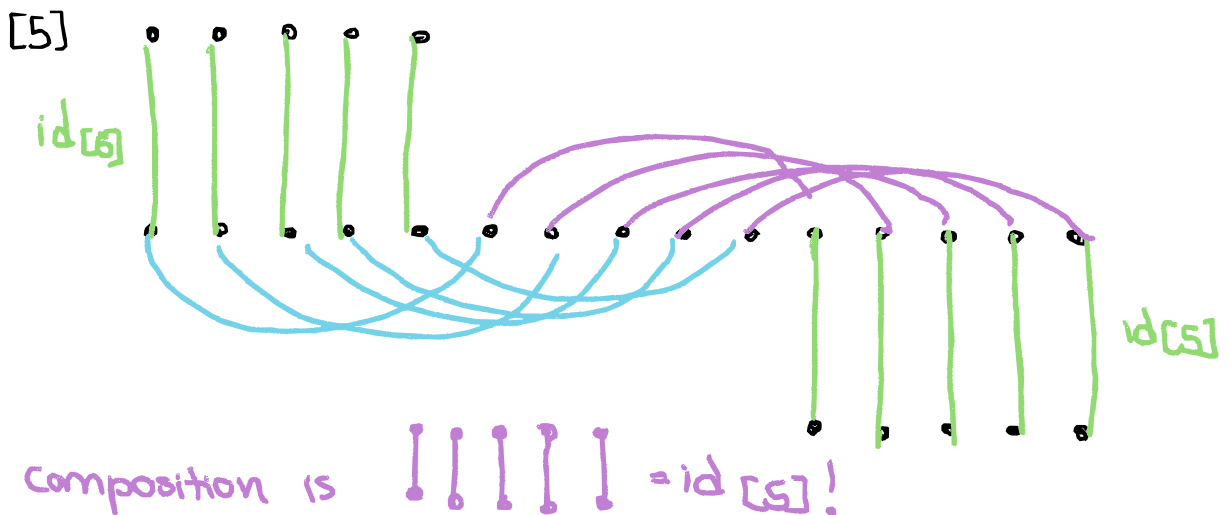
↳ unit is $[0]$, no points.

This is rigid, with elements being self dual!

Need: $\text{ev}: [n] \otimes [n] \rightarrow [0]$, $\text{coev}: [0] \rightarrow [n] \otimes [n]$



Need to check: $X \xrightarrow{1 \otimes \text{ev}} X \otimes X \otimes X \xrightarrow{\text{coev} \otimes 1} X = \text{id}_X \rightarrow X$
 $n=5$ just because



You can do a lot of proofs by diagram,
 including checking properties like $\square$, $\square$ axioms

We can also use it to find traces & dimensions!

$\dim [1] = ?$ Recall: $[0] \xrightarrow{cv} [1] \otimes [1] \xrightarrow{ev} [0]$
gives $\dim (1 \otimes 1)$

(no points)



$\leadsto$ composition diagram is trivial map
 $[0] \rightarrow [c]$, multiplied by $t^1 = t$.
so, $\dim [1] = t$

In $\text{Rep}(S_{-\pi})$, $\dim [1] = -\pi$

Remark: To get Deligne's category, we need to now take envelopes:

$$\widetilde{\text{Rep}}(S_t) \xleftarrow[\text{envelope}]{\text{karoubian}} \widehat{\text{Rep}}(S_t) \xleftarrow[\text{envelope}]{\text{additive}} \text{Rep}(S_t)$$

Remark: Mapping property for $\text{Rep}(S_n)$ which morally says $\text{Rep}(S_n)$ is a universal category for n -dim reps of finite groups extends to $\widetilde{\text{Rep}}(S_t)$, in a suitable setting.

Thm($\text{Rep } S_n$): let $\mathcal{T}$ be tensor category, $T \in \text{Ob}(\mathcal{T})$ be an n -dimensional algebra with $\wedge^{n+1} T = 0$ so that the pairing $x \otimes y \mapsto \text{Tr}(xy)$ is nondegenerate. Suppose that T also has the action of a finite group G . Then, we have an essentially unique functor $\mathcal{F}: \text{Rep}(S_n) \rightarrow \mathcal{T}$ with $\mathcal{F}(V) \cong T$ as G -algebras.

Thm(Rep(St)): let $\mathcal{T}$ be symmetric monoidal cat and $T \in \text{Ob}(\mathcal{T})$ satisfying the properties:

- T algebra
- T is self dual via $\text{Hom}(T \otimes T, 1) \simeq \text{Hom}(T, T^*)$,
- $T \otimes T \xrightarrow{\mu} T \xrightarrow{\text{tr}} 1$ non-degenerate
- $\dim T = t$

Then, there is an essentially unique additive symmetric monoidal functor $\text{Rep}(St) \rightarrow \mathcal{T}$, $[1] \mapsto T$

Rmk: $[n] \rightarrow V^{\otimes n}$, maps just follow the same rules but in a more generalized version

$\text{Rep}(GL_t) \leadsto$ very similar idea!

V standard representation of GL_n $\left\{ \begin{array}{l} \text{faithful rep} \\ g \mapsto \rho(g) \text{ injective} \end{array} \right.$
 $(\mathbb{C}^n \text{ with the usual action})$

No longer true $V^{\otimes m}$ has everything as a summand, but mixed products $V^{\otimes m} \otimes (V^*)^k$ do!

Schur-Weyl duality: $\overbrace{V \otimes V \otimes \dots \otimes V}^k$, $\dim V = n$.

$GL_n \curvearrowright$ coordinate-wise, $S_k \curvearrowright$ permuting tensor factors

$\Rightarrow$ These actions commute and generate everything

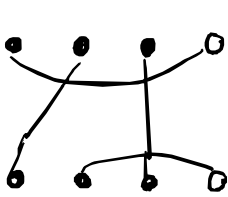
$\leadsto$ so if you want GL_n equivariant, that leaves the permuting

$\rightarrow$ Maps $V^{\otimes m} \rightarrow V^{\otimes m}$ permute entries

$\rightarrow$ Maps $(V^*)^{\otimes k} \rightarrow (V^*)^{\otimes k}$ permute entries

Special new case! Can pair up V, V^* using ev , and then put them back via $coev$:

use black dots for V , white for V^* .



$$V^{\otimes 3} \otimes V^*$$

$$V^{\otimes 3} \otimes V^*$$

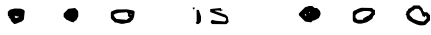
$$e_1, e_2, \dots, e_n \text{ basis } V$$
$$e_1^*, e_2^*, \dots, e_n^* \text{ basis } V^*$$

$$e_1 \otimes e_2 \otimes e_3 \otimes e_j^* = \delta_{ij} \sum_k e_2 \otimes e_k \otimes e_3 \otimes e_k^*$$

Composition $\leadsto$ new map obtained in a similar way as $\text{Rep}(St)$... We get $\dim V$ from connected components, replace by t .

Now the dual of

dev: (nothing)



Walled Brauer Category

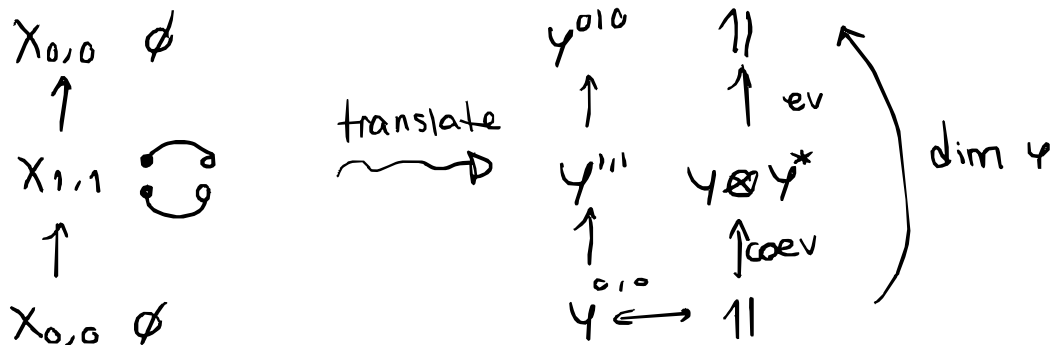
$$\left(\begin{array}{c} \text{Tensor functors} \\ \text{Rep}(GL_t) \rightarrow \mathcal{C} \end{array} \right) \cong \left(\begin{array}{c} \text{objects of } \mathcal{C} \\ \text{of dim } t \end{array} \right)$$

↓ 'nice enough'

Map $X_{1,0}$ to Y , $X_{n,m}$ to $Y^n \otimes (Y^*)^m$

Maps follow rules: vertical lines come from symmetry
horizontal from ev, co-ev.

key example:



$$\text{comp} = t \cdot \text{id}_{X_{0,0}}$$

compatible because $\dim Y = t$

My research: Snowden, Harman tried looking at $\text{Rep}(S_t)$ and see what was happening behind the construction

- Sort of limiting/stabilizing behavior $\leadsto$ should be associated to action by S_{∞} somehow
- Try to define 'permutation category' of S_{∞}
 - objects 'vector spaces which are equivariant'
 - morphisms 'matrices which are equivariant'
 - composition 'matrix multiplication' $\int A(x,y) B(y,z) dy$
measure?

Finding these measures is a very combinatorial problem, sometimes they don't exist, other times a family with a parameter exists (t from $\text{Rep}(S_t)$), other times there are like 4. Different measures give quite diff categories sometimes, cool examples were found,